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Projective representations of k-Galilei group

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Abstract. The projective representations of the k-Galilei group G_k are found by contracting the relevant representations of the κ-Poincaré group. The projective multiplier is found. It is shown that it is not possible to replace the projective representations of G_k by vector representations of some of its extensions.

1. Introduction

Recently attention has been paid to particular deformations of spacetime symmetries depending on a dimensionful parameter, the so-called κ-symmetries [1–12]. They are interesting because they provide a rather mild deformation of classical spacetime symmetries with a dimensionful parameter (cut-off?) naturally built in. It is, of course, still an open question whether quantum symmetries provide a proper way of introducing a fundamental energy/length scale into the theory; in particular, special attention should be paid to the problems related to non-cocommutativity of the coproduct which apparently seems to be in some contradiction with kinematical properties of many-particle systems (see, however, [8]). In spite of this, it could be interesting to study in more detail the properties of κ-deformed spacetime symmetries. Some preliminary studies of their physical implications have already been undertaken. In particular, Bacry [9] has found that they possess some attractive features from the point of view of the general requirements imposed on kinematical symmetries.

In most papers that have appeared so far the deformations of Poincaré symmetry have been studied. However, it seems to be interesting to analyse the deformation of its non-relativistic counterpart, i.e. the deformed Galilei group. One version of deformed Galilei algebra was studied in [13], where it was shown to provide some properties of the XXZ magnetic Heisenberg model, in particular the energy spectrum of the multi-quasi-particles bound states. In [3] another deformed Galilei algebra (the so-called k-Galilei algebra) was found by applying the contraction procedure ($c \to \infty$, $\kappa \to 0$, $k \equiv \kappa c$ kept fixed) to the κ-Poincaré algebra in a trigonometric version. The properties of this algebra (in a hyperbolic version) as well as the algebra obtained by letting $c \to \infty$, $\kappa \to \infty$, $k \equiv \kappa/c$ fixed, were studied in [7]. In [10], the κ-Poincaré group was contracted to the k-Galilei group and the latter was shown to be dual to the k-Galilei algebra. The bicross-product structure of both was revealed and the projective representations of the two-dimensional counterpart of the k-Galilei group were constructed. Other deformed Galilei algebras were constructed using a very general contraction scheme in [11] and [12], where their bicross-product structures were also exhibited.

In the present paper we continue the study of the k-Galilei group. In section 2 the projective multiplier is found by contracting the trivial multiplier on the κ-Poincaré group. In the $k \to \infty$ limit it reduces to the standard non-trivial multiplier on the classical Galilei group. In section 3 the unitary projective representations of the k-Galilei group are constructed (again by contraction from the representations of the κ-Poincaré group) and their infinitesimal form is given. The generators of the infinitesimal representations form the algebra which is a 'central' extension of the k-Galilei algebra. A question arises whether this structure can be lifted to the Hopf algebra structure. This question is equivalent to asking whether there exists a 'central' extension of the k-Galilei group such that the projective representations of the latter are equivalent to the standard (vector) representations of its central extension. In section 4 we prove that no such central extension exists. This was already noticed in [7], where it was shown that the relevant central extension cannot be obtained by a standard contraction procedure from the trivially extended κ-Poincaré group. However, it should be stressed that our result refers to the particular deformation of the Galilei group we are considering, for other deformations the central extension may exist (cf [7, 8]). Section 5 is devoted to some conclusions. Finally, the technicalities are relegated to the appendix.

We conclude the introduction with a short résumé of results obtained in [10].

In order to find the k-Galilei group one can apply the contraction procedure to the κ-Poincaré group defined in [2]. The following convenient parametrization of the Lorenz group can be used for the contraction procedure (in fact, it differs slightly from the one adopted in [2])

$$
\begin{aligned}
&\Lambda^0{}_0 = \frac{1}{\sqrt{1-\boldsymbol{v}^2/c^2}} \equiv \gamma \\
&\Lambda^0{}_i = \frac{\gamma}{c} v^i \\
&\Lambda^i{}_0 = \frac{\gamma}{c} v^i \\
&\Lambda^i{}_j = \left(\delta^i_k + (\gamma - 1)\frac{v^i v^k}{\boldsymbol{v}^2} \right) R^k{}_j \\
&R R^{\mathrm{T}} = I
\end{aligned}
\tag{1}
$$

as well as

$$
a^0 = c\tau.
$$

The following Hopf algebra G_k (k-Galilei group) is obtained from the contraction $\kappa \to 0$, $c \to \infty$, $k \equiv \kappa c$ fixed:

$$
\begin{aligned}
&[R^i{}_j, R^k{}_l] = 0 \qquad [R^i{}_j, v^k] = 0 \qquad [v^i, v^j] = 0 \\
&[a^i, a^j] = 0 \qquad [\tau, a^i] = \frac{\mathrm{i}}{k} a^i \\
&[\tau, v^i] = \frac{\mathrm{i}}{k} v^i \qquad [\tau, R^i{}_j] = 0 \\
&[v^i, a^j] = \frac{\mathrm{i}}{k}\left(\frac{1}{2}\boldsymbol{v}^2 \delta_{ij} - v^i v^j \right) \\
&[R^i{}_j, a^k] = \frac{\mathrm{i}}{k}(\delta_{ik} v^m R^m{}_j - v^i R^k{}_j) \\
&\Delta R^i{}_j = R^i{}_k \otimes R^k{}_j \\
&\Delta v^i = R^i{}_j \otimes v^j + v^i \otimes I
\end{aligned}
\tag{2}
$$

$$\Delta a^i = R^i{}_j \otimes a^j + v^i \otimes \tau + a^i \otimes I$$
$$\Delta\tau = \tau \otimes I + I \otimes \tau$$
$$(R^i{}_j)^* = R^i{}_j \qquad (v^i)^* = v^i \qquad (a^i)^* = a^i \qquad \tau^* = \tau.$$

G_k has a bicross-product structure

$$G_k = T^* \triangleright\!\blacktriangleleft C(E(3))$$

where $C(E(3))$ is the algebra of functions on the classical group $E(3)$ generated by R^i_j and v^i while T^* is defined by

$$[\tau, a^i] = \frac{\mathrm{i}}{k} a^i \qquad [a^i, a^j] = 0$$
$$\Delta a^i = a^i \otimes I + I \otimes a^i \qquad \Delta\tau = \tau \otimes I + I \otimes \tau.$$

The k-Galilei algebra $\widetilde{G_k}$, dual to G_k, reads

$$\begin{aligned}
&[J_i, J_k] = \mathrm{i}\epsilon_{ikl} J_l \qquad [J_i, K_k] = \mathrm{i}\epsilon_{ikj} K_j \qquad [J_i, P_k] = \mathrm{i}\epsilon_{ikj} P_j \\
&[K_i, H] = \mathrm{i}P_i \qquad [K_i, P_j] = \frac{\mathrm{i}}{2k}\delta_{ij}\boldsymbol{P}^2 - \frac{\mathrm{i}}{k} P_i P_j \\
&\Delta J_i = J_i \otimes I + I \otimes J_i \\
&\Delta H = H \otimes I + I \otimes H \\
&\Delta K_i = I \otimes K_i + K_i \otimes \mathrm{e}^{-H/k} - \frac{\mathrm{i}}{k}\epsilon_{ijk} J_i \otimes P_k \\
&\Delta P_i = I \otimes P_i + P_i \otimes \mathrm{e}^{-H/k} \\
&P_i^* = P_i \qquad H^* = H \qquad K_i^* = K_i \qquad J_i^* = J_i.
\end{aligned} \tag{3}$$

$\widetilde{G_k}$ has also a bicross-product structure

$$\widetilde{G_k} = T \blacktriangleright\!\triangleleft U(J, K)$$

where $U(J, K)$ is the universal covering of the Lie algebra $e(3)$ while T is defined by

$$[H, P_i] = 0 \qquad [P_i, P_j] = 0$$
$$\Delta H = H \otimes I + I \otimes H \qquad \Delta P_i = P_i \otimes \mathrm{e}^{-H/k} + I \otimes P_i.$$

The duality rules are the same as in classical case.

2. Projective multiplier on $\mathbf{G}_k$

In analogy with the classical case one can define the projective representation of a quantum group A acting on a Hilbert space $\mathcal{H}$ as a map ρ: $\mathcal{H} \to \mathcal{H} \otimes A$ satisfying

$$(\rho \otimes I) \circ \rho(\psi) = (I \otimes \omega)((I \otimes \Delta) \circ \rho(\psi)) \tag{4}$$

where ω is a unitary element of $A \otimes A$ (projective multiplier) obeying a suitable consistency condition [10].

Two projective representations ρ and ρ' are called equivalent if there exists a unitary element $\zeta \in A$ such that

$$\tilde{\rho} = (I \otimes \zeta)\rho. \tag{5}$$

The corresponding multipliers are related by the formula

$$(\zeta \otimes \zeta)\omega = \omega' \Delta(\zeta). \tag{6}$$

Obviously, a multiplier ω is trivial (the representation is equivalent to the vector one) if

$$\omega = (\zeta^{-1} \otimes \zeta^{-1})\Delta(\zeta). \tag{7}$$

In the classical case it is sometimes possible to obtain a non-trivial multiplier by contraction [14]. Assume the group $\hat{G}$ is obtained from G by contraction. Even if G does not admit non-trivial projective multipliers, one can proceed as follows. Let $\zeta(g)$ be a unitary function on G, $\zeta(g)\zeta^*(g) = 1$. Define a trivial multiplier on G

$$\omega(g, g') = \zeta^*(g)\zeta^*(g')\zeta(gg'). \tag{8}$$

It can happen that, $\zeta(g)$ being properly chosen, the specific combination of ζ's appearing on the right-hand side of equation (8) tends to a well-defined limit under contraction while $\zeta(g)$ itself has no such limit. We can then expect that the limiting $\omega(g, g')$ is a non-trivial multiplier on $\hat{G}$. This is, for example, the case for G being the Poincaré group and

$$\zeta(\{\Lambda, a\}) = \mathrm{e}^{-\mathrm{i}mca^0}. \tag{9}$$

The corresponding multiplier $\omega(g, g')$, equation (8), gives in the contraction limit $c \to \infty$

$$\tilde{\omega} = \mathrm{e}^{-\mathrm{i}m((v^2/2)\tau' + v^k R^k{}_i a'^i)} \tag{10}$$

the standard multiplier on the Galilei group.

Following the classical case we define the trivial projective multiplier on the κ-Poincaré group:

$$\omega = (\mathrm{e}^{\mathrm{i}mca^0} \otimes \mathrm{e}^{\mathrm{i}mca^0})\, \mathrm{e}^{-\mathrm{i}mc(\Lambda^0{}_\mu \otimes a^\mu + a^0 \otimes I)}. \tag{11}$$

Our aim is to find the limiting form of ω for $\kappa \to 0$, $c \to \infty$, $k \equiv \kappa c$; the κ, c-dependence of m is unknown and must be determined from the condition that the non-trivial limit exists. To this end, we first rewrite ω in a more convenient form making explicit the cancellation of divergent terms. The long and rather tedious calculations reported in the appendix lead to the following expression for ω:

$$\begin{aligned}\omega = {}& \exp[\mathrm{i}(mc - \kappa \ln(\cosh(mc/\kappa) + \Lambda^0_0 \sinh(mc/\kappa))) \otimes a^0] \\ & \times \exp\left[-\mathrm{i}\kappa \frac{\sinh(mc/\kappa)\Lambda^0{}_k}{\cosh(mc/\kappa) + \Lambda^0{}_0 \sinh(mc/\kappa)} \otimes a^k\right].\end{aligned} \tag{12}$$

In order to calculate the limiting value of ω we use the parametrization (1). It is easy to check that in order to obtain the non-trivial limit one can choose the following form of m

$$m = \frac{-k}{2c^2} \ln\left(1 - \frac{2Mc^2}{k}\right) \tag{13}$$

where M is some fixed mass parameter (which we assume to be positive); let us note that for k negative, $m > 0$ and $m \to 0$ for $c \to \infty$ while $mc^2\tau$ diverges. Some remarks are here in order. The form of formula (13) is almost uniquely (up to the terms that vanish in the contraction limit) determined by the condition that the limiting form of equation (12) exists. Contrary to the 'undeformed' case the non-relativistic mass M is not equal to its relativistic counterpart m; this property is, however, restored in the $k \to \infty$ limit. On the other hand, the relation between m and M is purely classical, i.e. it does not contain the Planck constant. The Planck constant appears explicitly in the exponents on the right-hand side of equation (11). Consequently, it also appears in the same way in equation (12); this is in agreement with the fact that κ has the dimension of the inverse of momentum. It follows then that k has the dimension of the inverse of energy and equation (13) does not contain the Planck constant.

Taking the $c \to \infty$ limit in equation (12) one obtains

$$\tilde{\omega} = \exp\left[-\mathrm{i}k \ln\left(1 + \frac{M\boldsymbol{v}^2}{2k}\right) \otimes \tau\right] \exp\left[-\frac{Mv^k R^k{}_i}{1 + (M\boldsymbol{v}^2/2k)} \otimes a^i\right] \tag{14}$$

which can also be written as

$$\tilde{\omega} = \exp\left[-\mathrm{i}\left(\frac{2k}{M\boldsymbol{v}^2}\ln\left(1+\frac{M\boldsymbol{v}^2}{2k}\right)\otimes I\right)\left(\frac{M\boldsymbol{v}^2}{2}\otimes\tau + Mv^kR^k{}_i\otimes a^i\right)\right]. \tag{15}$$

This expression is a natural generalization of the one obtained in [10] for the two-dimensional case. In the classical limit $k \to \infty$ it coincides with the standard multiplier (10).

Let us conclude this section by noting one problem related to formula (14). In order to keep the Poincaré mass m real we had to assume k negative. This, however, implies that $\tilde{\omega}$ is singular somewhere. On the other hand, with k positive, $\tilde{\omega}$ is everywhere regular.

It seems that this problem cannot be cured in a simple way. In the following section we will present an argument that supports this point of view.

3. Contraction of representations

The unitary representations of the κ-Poincaré group were constructed in [15] (see also [16]). This construction can be summarized as follows. The representation space is the Hilbert space of square integrable (with respect to the standard measure $\mathrm{d}^3\boldsymbol{p}/2p_0$) functions over the hyperboloid $p^2 = m^2$ taking their values in the vector space carrying the spin s representation of the rotation group (s is assumed to be integer, for s half-integer one should consider the quantum $\mathrm{ISL}(2,\mathbb{C})$ group [17] which only amounts to small modifications). The (right) corepresentation reads

$$\begin{aligned}\rho : f_i(p_\mu) \to \exp&\left[-\mathrm{i}\kappa\ln(\cosh mc/\kappa) + \frac{p_0}{mc}\sinh(mc/\kappa)\otimes a^0\right]\\ &\times\exp\left[\frac{-\mathrm{i}\kappa\sinh(mc/\kappa)p_k}{mc\cosh(mc/\kappa)+p_0\sinh(mc/\kappa)}\otimes a^k\right]\\ &\cdot D_{ij}(R(p\otimes I, I\otimes\Lambda))f_j(p_\nu\otimes\Lambda^\nu{}_\mu).\end{aligned} \tag{16}$$

Here by $D(R(p\otimes I, I\otimes\Lambda))$ we denote the spin s representation of the standard Wigner rotation written as an element of the tensor product $\mathcal{H}\otimes A$.

It follows from equation (16) that the whole deformation is contained in the translation sector; in other words, the representation is obtained by integrating the infinitesimal representation given in the Majid–Ruegg basis [18, 19]. In the limit $\kappa \to \infty$ unitary representations of the classical Poincaré group are recovered.

In order to find the representations of the k-Galilei group we apply again the contraction procedure. To this end we put $\kappa = k/c$ and take $m \equiv m(M, k, c)$ as defined by equation (13). As in the classical case it is necessary to subtract the rest energy by redefining ρ:

$$\tilde{\rho} \equiv (I\otimes \mathrm{e}^{-\mathrm{i}mca^0})\rho. \tag{17}$$

Finally, in contrast to the classical case, we have to redefine the momenta and the wavefunctions as follows:

$$\boldsymbol{p} = \frac{m}{M}\boldsymbol{q} \qquad \frac{m}{M}f_i(\boldsymbol{p}) = \tilde{f}_i(\boldsymbol{q}). \tag{18}$$

It is now easy to check that the limit $c \to \infty$ exists and gives the following unitary representation of the k-Galilei group:

$$\begin{aligned}\widetilde{\rho_{nr}} : \tilde{f}_i(\boldsymbol{q}) \to \exp&\left[-\mathrm{i}k\ln\left(1+\frac{\boldsymbol{q}^2}{2Mk}\right)\otimes\tau\right]\\ &\times\exp\left[\frac{-\mathrm{i}q_k}{1+\boldsymbol{q}^2/2Mk}\otimes a^k\right](I\otimes D_{ij}(R))\tilde{f}_j(q_k\otimes R^k{}_i + I\otimes Mv^kR^k{}_i)\end{aligned} \tag{19}$$

acting on the Hilbert space of square integrable functions with respect to the invariant measure $d^3\boldsymbol{q}$ and taking values in the vector space carrying the spin s representation of the rotation group.

Using the results contained in [20] one easily concludes that the general form of the irreducible (co)representation of the κ-Poincaré group can be obtained by replacing the exponentials on the right-hand side of equation (16) by

$$\exp\left[-\mathrm{i}\kappa\ln\left(\frac{p_0+C}{A}\right)\otimes a^0\right]\exp\left[-\frac{\mathrm{i}\kappa p_k}{p_0+C}\otimes a^k\right]$$

where $A = A(m, c, \kappa)$, $C = C(m, c, \kappa)$ are two real functions subject to the condition $C^2 - A^2 = m^2c^2$, but otherwise arbitrary. So the question arises whether our trouble can be cured by an appropriate choice of A and C such to obtain, in the $c \to \infty$, $\kappa \to 0$ limit, the representation given by formula (19) while $m = m(M, c, k)$ lies, for $k > 0$, in the physical region $m > 0$. This seems not to be possible. Let us put again $\boldsymbol{p} = (m/M)\boldsymbol{q}$ and consider the second exponential. It is easy to see that, in order to obtain the proper limiting formula, the following condition should be fulfilled

$$\lim_{c\to\infty} c^2\left(1+\frac{C}{mc}\right) = \frac{k}{M}.$$

However, due to the condition $C^2 - A^2 = m^2c^2$, $|C| \geqslant mc$ and the above equation can only be satisfied provided $C = -mc - \Delta$, $\Delta \geqslant 0$. Then

$$\lim_{c\to\infty} \frac{c\Delta}{m} = -\frac{k}{M}$$

which is impossible for $k > 0$, $M > 0$ and $m > 0$.

As a next step let us find the infinitesimal form of the representation $\widetilde{\rho_{nr}}$. Let us recall that if

$$\rho : \mathcal{H} \ni f \to f_{(\alpha)} \otimes a_{(\alpha)} \in \mathcal{H} \otimes A$$

is the (right) corepresentation of the quantum group A then any element X of the dual Hopf algebra (quantum Lie algebra) is represented by the operator

$$\tilde{X} : \mathcal{H} \ni f \to f_{(\alpha)} < a_{(\alpha)}, X > \in \mathcal{H}. \tag{20}$$

The relevant duality rules can be, as was mentioned above, adopted from classical theory. A simple calculation then gives

$$\begin{aligned}
J_k &= -\mathrm{i}\epsilon_{klm} q_l \frac{\partial}{\partial q_m} + s_k \\
K_k &= \mathrm{i}M \frac{\partial}{\partial q_k} \\
H &= k\ln\left(1+\frac{\boldsymbol{q}^2}{2Mk}\right) \\
P_k &= \frac{q_k}{1+(\boldsymbol{q}^2/2Mk)}.
\end{aligned} \tag{21}$$

Let us note that H and P_k are non-singular only provided $k > 0$.

The operators (21) verify the following commutation rules:

$$\begin{aligned}
&[J_i, J_k] = \mathrm{i}\epsilon_{ikl} J_l \qquad [J_i, P_k] = \mathrm{i}\epsilon_{ikl} P_l \\
&[J_i, K_k] = \mathrm{i}\epsilon_{ikl} K_l \qquad [K_i, H] = \mathrm{i}P_i \\
&[K_i, P_j] = \mathrm{i}M\delta_{ij}\,\mathrm{e}^{-2H/k} + \frac{\mathrm{i}}{2k}\delta_{ij}\boldsymbol{P}^2 - \frac{\mathrm{i}}{k}P_iP_j.
\end{aligned} \tag{22}$$

For $M \to 0$ this algebra coincides with the algebraic sector of the k-Galilei algebra (3).

Finally, let us note the following dispersion relation which is valid within the representation (21):

$$k(1 - \mathrm{e}^{-H/k}) = \frac{\boldsymbol{P}^2}{2M}. \tag{23}$$

4. Central extension of the k-Galilei group

It is well known that in the classical case, given a projective representation of a group G, one can construct the group G' such that this projective representation of G is equivalent to the vector representation of G'. A natural question arises whether the analogous construction is possible in the quantum case.

Let us assume that there exists a Hopf algebra G'_k such that:

(i) G'_k is obtained from G_k by adding one new unitary element $\zeta : \zeta\zeta^* = \zeta^*\zeta = I$;

(ii) G_k is a Hopf subalgebra of G'_k;

(iii) $\Delta(\zeta) = (\zeta \otimes \zeta)\omega$ where ω is a projective multiplier.

Then, if ρ is a projective representation of G_k determined by ω (cf equation (4)),

$$\rho' = (I \otimes \zeta)\rho$$

is a vector representation of G'_k.

In the commutative case (iii) determines G'_k uniquely and consistently. In the quantum case, however, Δ should be a homomorphism which, together with (iii), imposes non-trivial consistency conditions. It has been already shown [7] that G'_k cannot be obtained by a straightforward generalization of a standard contraction from a trivial extension of κ-Poincaré. We show below that there is no solution to the problem, at least if the existence of a well-defined limit $k \to \infty$ which reproduces the classical situation is assumed. To this end, let us note first that equations (2) and (14) imply

$$\begin{aligned}&\exp[\alpha(\tau \otimes I + I \otimes \tau)]\,\tilde{\omega}\exp[-\alpha(\tau \otimes I + I \otimes \tau)]\\ &\quad= \exp\left[\mathrm{i}\left(\frac{2k}{M\boldsymbol{v}^2}\ln\left(1 + \frac{M\boldsymbol{v}^2}{2k}\,\mathrm{e}^{2\mathrm{i}\alpha/k}\right)\otimes I\right)\left(\frac{M\boldsymbol{v}^2}{2}\otimes\tau + Mv^kR^k{}_i \otimes a^i\right)\right]\end{aligned} \tag{24}$$

therefore,

$$[\tau \otimes I + I \otimes \tau, \tilde{\omega}] = \frac{2}{k}\tilde{\omega}(M\boldsymbol{v}^2 \otimes \tau + Mv^kR^k{}_i \otimes a^i)\left(\frac{1}{1 + (M\boldsymbol{v}^2/2k)} \otimes I\right). \tag{25}$$

The homomorphism condition

$$[\Delta(\zeta), \Delta(\tau)] = \Delta([\zeta, \tau]) \tag{26}$$

gives

$$\begin{aligned}&\Delta(\zeta)\left(\frac{2}{k}\left(\frac{1}{1 + (M\boldsymbol{v}^2/2k)} \otimes I\right)\left(\frac{M\boldsymbol{v}^2}{2}\otimes\tau + Mv^kR^k{}_i \otimes a^i\right)\right)\\ &\qquad + (I \otimes \zeta)([\zeta, \tau] \otimes I)\tilde{\omega} + (\zeta \otimes I)(I \otimes [\zeta, \tau])\tilde{\omega} = \Delta([\zeta, \tau]).\end{aligned} \tag{27}$$

It follows from equation (27) that the commutator $[\zeta, \tau]$ should be of the form

$$[\zeta, \tau] = \frac{2}{k}\zeta X \tag{28}$$

where X is an element of G_k, the $1/k$ factor is extracted out explicitly and the factor 2 is written for convenience.

The following relation follows immediately from equations (27) and (28):

$$\Delta(X) = \tilde{\omega}^{-1}(X \otimes I)\tilde{\omega} + \tilde{\omega}^{-1}(I \otimes X)\tilde{\omega} + \left(\frac{1}{1+(M\boldsymbol{v}^2/2k)} \otimes I\right) \times \left(\frac{M\boldsymbol{v}^2}{2} \otimes \tau + Mv^k R^k{}_i \otimes a^i\right). \tag{29}$$

Taking the lowest term in the $1/k$ expansion we get

$$\frac{M\boldsymbol{v}^2}{2} \otimes \tau + Mv^k R^k{}_i \otimes a^i = \Delta(X) - X \otimes I - I \otimes X \tag{30}$$

which can be viewed as the relation on the classical Galilei group. However, equation (30) does not hold true because the left-hand side is not a coboundary.

5. Conclusions

Using the contraction technique we have found the projective multipliers on the k-Galilei group and the corresponding projective representations, both in global as well as in infinitesimal form. It appears that we obtain a well-defined and regular structure for $c \to \infty$ provided the deformation parameter k is taken to be positive. On the other hand, in order to keep the Poincaré mass parameter in the allowed region, in the course of contraction we should rather assume k to be negative. We do not have a clear understanding of this phenomenon.

In the classical case the projective representations can always be converted into the vector representations of a suitably defined extension of the original group. We have seen in section 4 that this is not necessarily the case for quantum groups. There exists no suitable extension of G_k which, in the classical limit $k \to \infty$ reduces to the standard case. From the physical point of view this seems to be rather a technical and not a very serious obstacle because we are finally interested in the projective representations of the original (quantum) group. On the other hand, the problem seems to be mathematically interesting. In the classical case, the equivalence between the projective representations and the two–cocycle extensions allows us to reduce the theory of such representations to the standard theory of vector representations. Our result suggests that in the deformed case the two-cocycle extensions and the projective representations might be non-interchangeable concepts. However, this problem calls for a deeper study which should be based on a general theory of two-cocycle extensions of Hopf algebras [21, 22].

A problem which certainly deserves further study is the multiplication of representations. This is important if we would like to reconcile the non-cocommutativity of the algebra coproduct with the basic properties of many-particle systems, especially those containing identical particles.

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Appendix

We derive here equation (12). In order to simplify the notation we omit the tensor product symbol $\otimes$, writing instead with a prime the factors appearing on the right side of it.

Equation (11) then reads

$$\omega = \mathrm{e}^{\mathrm{i}mca^0}\,\mathrm{e}^{\mathrm{i}mca'^0}\,\mathrm{e}^{-\mathrm{i}mc(\Lambda^0{}_0a'^0+\Lambda^0{}_ka'^k+a^0)}. \tag{A.1}$$

Let us define

$$X(m) \equiv \mathrm{e}^{\mathrm{i}mca^0}\,\mathrm{e}^{-\mathrm{i}mc(\Lambda^0{}_0a'^0+\Lambda^0{}_ka'^k+a^0)}. \tag{A.2}$$

Then $X(0) = I$ and $X(m)$ obeys the equation

$$\dot{X}(m) = (-\mathrm{i}c)(Y_0(m)a'^0 + Y_k(m)a'^k)X(m) \tag{A.3}$$

where

$$Y_\mu(m) = \mathrm{e}^{\mathrm{i}mca^0}\Lambda^0{}_\mu\,\mathrm{e}^{-\mathrm{i}mca^0} \qquad Y_\mu(0) = \Lambda^0{}_\mu. \tag{A.4}$$

Let us first calculate $Y_0(m)$. Using the κ-Poincaré group commutation rules we get

$$\dot{Y}_0(m) = \mathrm{i}c\,\mathrm{e}^{\mathrm{i}mca^0}[a^0, \Lambda^0{}_0]\,\mathrm{e}^{-\mathrm{i}mca^0} = -\frac{c}{\kappa}\,\mathrm{e}^{\mathrm{i}mca^0}((\Lambda^0{}_0)^2 - 1)\,\mathrm{e}^{-\mathrm{i}mca^0}$$
$$= -\frac{c}{\kappa}(Y_0^2(m) - 1). \tag{A.5}$$

As the $Y_0(m)$ commute for all m, (A.5) together with the initial condition (A.4) can be solved by separation of variables yielding

$$Y_0(m) = \frac{\Lambda^0{}_0\cosh(mc/\kappa) + \sinh(mc/\kappa)}{\Lambda^0{}_0\sinh(mc/\kappa) + \cosh(mc/\kappa)}. \tag{A.6}$$

With $Y_0(m)$ explicitly known one can apply the same procedure to find $Y_k(m)$; the result reads

$$Y_k(m) = \frac{\Lambda^0{}_k}{\cosh(mc/\kappa) + \Lambda^0{}_0\sinh(mc/\kappa)}. \tag{A.7}$$

Now we can go back to equation (A.3). It cannot be solved directly because the two terms on the right-hand side do not commute. To account for this we pass to the 'interaction picture' and define

$$X(m) = \exp\left[-\mathrm{i}c\int_0^m \mathrm{d}m\,Y_0(m)a'^0\right]W(m) \qquad W(0) = I. \tag{A.8}$$

For (A.3) we get

$$\dot{W}(m) = -\mathrm{i}cY_k(m)\exp\left[\mathrm{i}c\int_0^m \mathrm{d}m\,Y_0(m)a'^0\right]a'^k\exp\left[-\mathrm{i}c\int_0^m \mathrm{d}m\,Y_0(m)a'^0\right]W(m). \tag{A.9}$$

On the other hand,

$$\mathrm{i}c\int_0^m \mathrm{d}m\,Y_0(m) = \mathrm{i}\kappa\ln(\cosh(mc/\kappa) + \Lambda^0{}_0\sinh(mc/\kappa)). \tag{A.10}$$

It follows from (A.9) and (A.10) that

$$\dot{W}(m) = \frac{-\mathrm{i}c\Lambda^0{}_ka'^k}{(\cosh(mc/\kappa) + \Lambda^0{}_0\sinh(mc/\kappa))^2}W. \tag{A.11}$$

Again everything commutes so that (A.11) can be solved

$$W = \exp\left[-\mathrm{i}c\int_0^m \frac{\mathrm{d}m}{(\cosh(mc/\kappa) + \Lambda^0{}_0\sinh(mc/\kappa))^2}\Lambda^0{}_ka'^k\right] \tag{A.12}$$

or

$$W = \exp\left[\mathrm{i}\kappa\frac{\sinh(mc/\kappa)\Lambda^0{}_ka'^k}{\cosh(mc/\kappa) + \Lambda^0{}_0\sinh(mc/\kappa)}\right]. \tag{A.13}$$

Equation (11) follows directly from (A.1), (A.2), (A.8) and (A.13).

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